APPENDIX A

A.1 TENSOR FIELDS

A field is a point function that describes the property of a material body. In general, a field is continuous no matter how small a volume of the body is considered. Discontinuity may occur in a field where there are shock waves, dislocations, etc. On either side of the discontinuity, the property of the field remains to be in effect. Some examples of fields are

(1) scaler fields--tensor of zeroth rank; mass density ρ, temperature T, etc.

(2) vector fields--tensor of first rank; displacement $\bar{u}$, velocity $\bar{v}$, acceleration $\bar{a}$, etc.

(3) tensor fields--tensor of second and higher ranks, stress $\tilde{\sigma}$, strain $\tilde{\epsilon}$, etc.

These fields can be denoted in Gibbs notation as $\rho, \bar{v}, \tilde{\sigma}$, etc., or in Cartesian tensor notation ρ, v_i, σ_{ij}, etc. The tensor of the nth rank carries "n" distinct subscripts and its components are properly-ordered 3^n-numbers.

Summation Convention, Special Tensors

(1) Summation convention: any repeated subscript denotes a sum over that subscript from 1 to 3 unless indicated otherwise. Some examples are

$$\text{(a) } A_{ii} = \sum_{i=1}^{3} A_{ii} = A_{11} + A_{22} + A_{33} \qquad \text{(A.1-1)}$$

(b) $A_{ij}B_j = \sum_{i=1}^{3} A_{ij}B_j = A_{i1}B_1 + A_{i2}B_2 + A_{i3}B_3$ (A.1-2)

The repeated indices reduce the rank of the tensor to one less and is called a contraction in the rank.

(2) Special tensors:

(a) identity tensor or isotropic tensor

$$\delta_{ij} = \begin{cases} 1 \text{ if } i=j \\ 0 \text{ if } i \neq j \end{cases} \qquad \text{(A.1-3)}$$

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3 \qquad \text{(A.1-4)}$$

(b) alternative tensor or permutation tensor

$$e_{ijk} = \begin{cases} +1 \text{ if } i,j,k \text{ is even permutation of } 1,2,3 \\ 0 \text{ if any indices repeated} \\ -1 \text{ if } i,j,k \text{ is odd permutation of } 1,2,3 \end{cases} \qquad \text{(A.1-5)}$$

$$e_{ijk}e_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp} \qquad \text{(A.1-6)}$$

$$e_{ijk}e_{pjk} = 2\delta_{ip} \qquad \text{(A.1-7)}$$

$$e_{ijk}e_{ijk} = 6 \qquad \text{(A.1-8)}$$

Tensor Operations: laws of tensor

(1) Algebra of tensor

(a) multiplication by a constant: $\alpha(A_{ij}) = \alpha A_{ij}$ (A.1-9)

(b) addition: $A_{ij} + B_{ij} = C_{ij}$ (A.1-10)

(c) multiplication: $A_{ij}B_k = C_{ijk}$ (A.1-11)

(d) contraction - make two subscripts the same:

$$A_{ij}B_j = C_i \qquad (A.1\text{-}12)$$

$$A_iB_i = \phi \qquad (A.1\text{-}13)$$

(e) isomers - by exchanging order of two subscripts, if tensor is symmetric in i and j then

$$A_{ijk...} = A_{jik...} \qquad (A.1\text{-}14)$$

and if tensor is anti-symmetric or skew-symmetric in i an j then

$$A_{ijk...} = -A_{jik...} \qquad (A.1\text{-}15)$$

(2) Calculus of tensor: differentiation and integration of tensors

(a) del operator $\bar{\nabla}$: $\bar{\nabla} = \bar{e}_i\partial/\partial x_i = \bar{e}_i\partial_i$ (A.1-16)

(b) grad $\phi = \bar{\nabla}\phi \rightarrow \partial_i\phi$ (A.1-17)

(c) div $\bar{v} = \bar{\nabla}\cdot\bar{v} \rightarrow \partial_i v_i$ or $v_{i,i}$ (A.1-18)

(d) curl $\bar{v} = \nabla \times \bar{v} \rightarrow e_{ijk}\partial_j v_k$ or $e_{ijk}v_{k,j}$ (A.1-19)

(e) $\nabla^2\phi = \bar{\nabla}\cdot\bar{\nabla}\phi \rightarrow \partial_{ii}\phi$ or $\phi_{,ii}$ (A.1-20)

(f) dual operation: if $\omega_i = e_{ijk}T_{jk}$, then (A.1-21)

$$T_{pq} = \omega_i e_{ipq}/2 = -T_{qp} \qquad \text{(A.1-22)}$$

(g) Stokes' Theorem, Fig.1.1-1.

$$\oint_c \bar{F}\cdot d\bar{x} = \iint_S \bar{n}\cdot(\bar{\nabla}\times\bar{F})\, dS \rightarrow \qquad \text{(A.1-23a)}$$

$$\oint_c F_i dx_i = \iint_S n_i\, e_{ijk}\, F_{k,j}\, dS \qquad \text{(A.1-23b)}$$

(h) Gauss' Theorem, Fig. A.1-1.

$$\iiint_V \partial\bar{T}/\partial\bar{x}\, dV = \iint_S \bar{T}\cdot\bar{n}\, dS \rightarrow \qquad \text{(A.1-24a)}$$

$$\iiint_V T_{ijk\ldots,p}\, dV = \iint n_p\, T_{ijk\ldots}\, dS \qquad \text{(A.1-24b)}$$

(i) Green's Theorem in two-dimension, Fig.A.1-2.

$$\int_c (Pdx+Qdy) = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

where D is simply connected and bounded by a closed contour c.

(j) Green's Theorem in three-dimension, Fig.A.1-3

$$\int_c (Pdx+Qdy+Rdz) = \int_\partial\int \{(\frac{\partial R}{\partial x} - \frac{\partial Q}{\partial z}) \cos(n,x)$$

$$+ (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}) \cos(n,y) + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) \cos(n,z)\}ds$$

where ∂ is simply connected region on a surface S bounded by a closed contour c.

*<u>Orthogonal Transformation, Orthogonality Conditions</u>: Fig.A.1-4

(1) Orthogonal transformation

$$x_i = f_i(x_1', x_2', x_3') \qquad i = 1, 2, 3 \qquad \text{(A.1-25)}$$

$$x_i' = g_i(x_1, x_2, x_3) \qquad i = 1, 2, 3 \qquad \text{(A.1-26)}$$

where f_i, g_i are single-valued, continuous functions and they possess continuous first partial derivatives in the region, and

$$J = |a_{ij}| = \det |a_{ij}| \neq 0 \qquad \text{(A.1-27)}$$

$$a_{ij} = \overset{j}{e_i} = \cos \alpha_{ij} = \cos (x_i, x_j') \qquad \text{(A.1-28)}$$

(2) Orthogonality conditions

$$a_{ij}\, a_{ik} = \delta_{jk} \qquad (A.1\text{-}29)$$

$$a_{ji}\, a_{ki} = \delta_{jk} \qquad (A.1\text{-}30)$$

<u>Law of Transformation for Cartesian Tensors</u>

(1) scaler: $\phi' = \phi$ (A.1-31)

(2) vector: $A'_j = a_{ij}A_i$ (A.1-32)

(3) second rank tensor: $B'_{pq} = a_{ip}a_{jq}B_{ij}$ (A.1-33)

(4) higher rank tensor: $C'_{pqr\ldots} = a_{ip}a_{jq}a_{kr}\ldots C_{ijk\ldots}$ (A.1-34)

<u>Second Rank Tensor:</u> $\tilde{T} \rightarrow T_{ij}$

(1) symmetry: $T_{ij}(\bar{x}) = T_{ji}(\bar{x})$, $\bar{x}$ = position vector (A.1-35)

(2) skew-symmetry: $T_{ij}(\bar{x}) = -T_{ji}(\bar{x})$ (A.1-36)

(3) invariants I, II, III of T_{ij}

$$I_T = T_{ii} \qquad (A.1\text{-}37)$$

$$II_T = (T_{ii}T_{jj} - T_{ij}T_{ji})/2 \qquad (A.1\text{-}38)$$

$$III_T = |T_{ij}| = \det T_{ij} \qquad (A.1\text{-}39)$$

(4) deviator tensor: $t_{ij} = T_{ij} - T_{kk}\delta_{ij}/3$ (A.1-40)

(5) spherical part of tensor: $T = (T_{xx} + T_{yy} + T_{zz})/3$ (A.1-41)

Second Rank Tensor: Principal Values, Principal Directions.

Consider

$$Q_i = A_{ij}P_j \tag{A.1-42}$$

Of all possible P_j, now seek the P_i that is in the same direction as Q_i, i.e.

$$Q_i = \lambda P_i \tag{A.1-43}$$

From Eq.(A.1-42) and Eq.(A.1-43) the following is obtained:

$$\lambda P_i = A_{ij}P_j \quad \text{or} \quad (A_{ij} - \lambda\delta_{ij})P_j = 0 \tag{A.1-44}$$

The vectors P_j satisfying this equation are called eigen vetors or principal vectors. For non-trivial solution the determinant must vanish:

$$\det(A_{ij} - \lambda\delta_{ij}) = \begin{vmatrix} A_{11}-\lambda & A_{12} & A_{13} \\ A_{21} & A_{22}-\lambda & A_{23} \\ A_{31} & A_{32} & A_{33}-\lambda \end{vmatrix} = 0 \tag{A.1-45}$$

Expanding the above we obtain the characteristic equation:

$$\lambda^3 - I\,\lambda^2 + II\,\lambda - III = 0 \tag{A.1-46}$$

where I, II, III are the invariants for the tensor A_{ij} as defined in Eqs.(A.1-37) to (A.1-39). The roots of Eq.(A.1-45) or (A.1-46) are called eigen values or principal values. For symmetric second rank tensor, the three roots are always real but not necessarily distinct and the corresponding eigen vectors are mutually orthogonal.

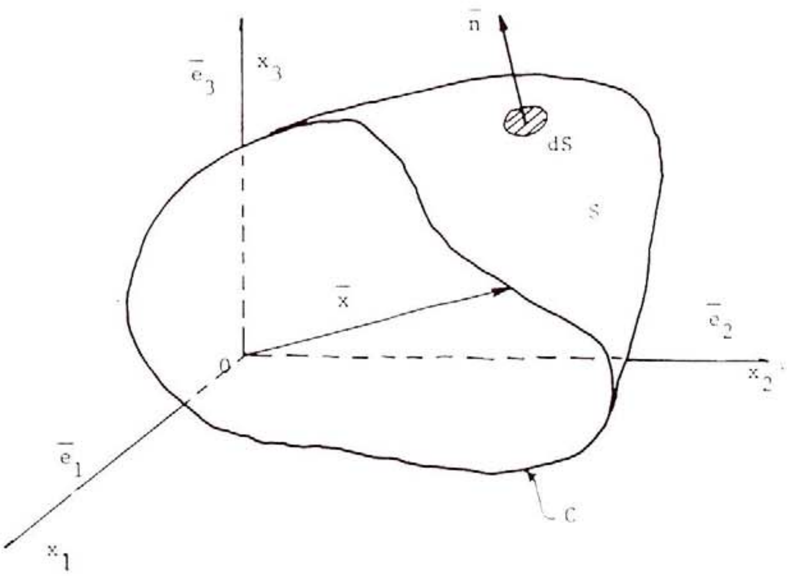

Fig.A.1-1 Gauss theorem

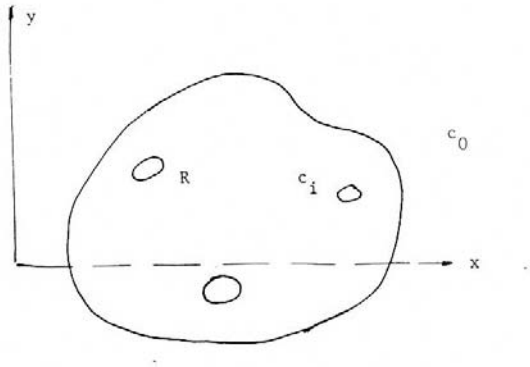

Fig.A.1-2 Green's theorem: two dimensions

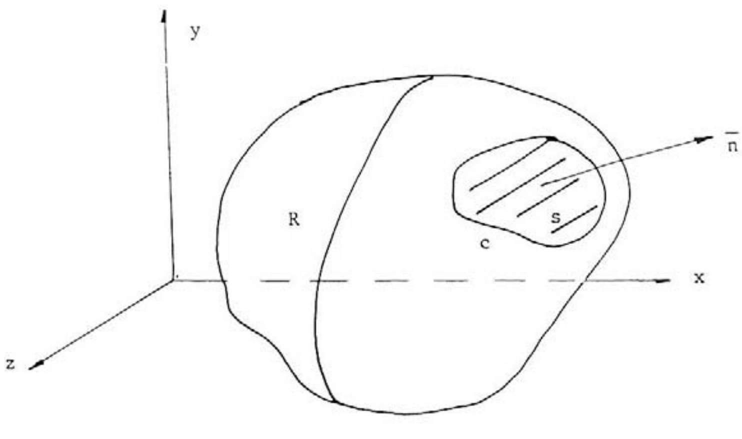

Fig.A.1-3 Green's theorem: three dimensions

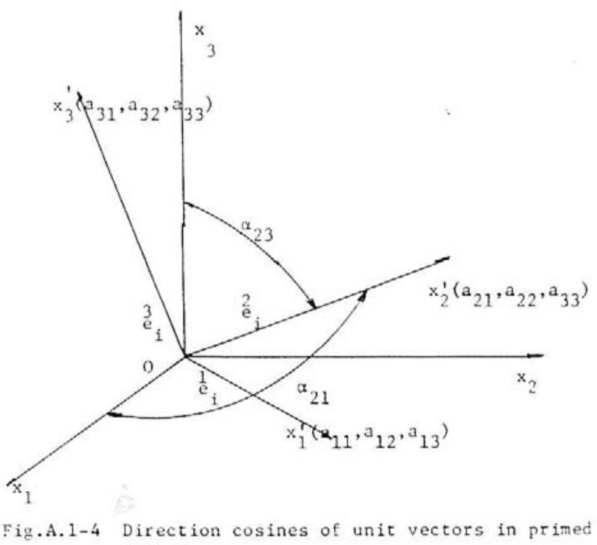

Fig.A.1-4 Direction cosines of unit vectors in primed system